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SECANT VARIETIES TO THE VARIETY OF REDUCIBLE FORMS

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Abstract. We completely classify the dimension of secant varieties $\mathrm{Sec}_1(\mathbb{X}_{\lambda,2})$ to the variety of reducible forms in $\Bbbk[x_0, x_1, x_2]$ when $\lambda = (1, \dots, 1, 3, \dots, 3)$, and also show that they are all non-defective.

1. Introduction

Let $R = \Bbbk[x_0, x_1, \dots, x_n]$ be an $(n+1)$-variable polynomial ring over a field $\Bbbk$ and let I be a homogeneous ideal of R (or the ideal of a subscheme in $\mathbb{P}^n$). Then the numerical function

$$\mathbf{H}(R/I, t) := \dim_\Bbbk R_t - \dim_\Bbbk I_t$$

is called a *Hilbert function* of the ring R/I. If $I := I_\mathbb{X}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote the Hilbert function of $\mathbb{X}$ by

$$\mathbf{H}_\mathbb{X}(t) := \mathbf{H}(R/I_\mathbb{X}, t).$$

To introduce a star-configuration, we start with varieties of some specific ideals of R. In [2], the authors proved that if $F_1, \dots, F_s$ are general forms in $R = \Bbbk[x_0, x_1, \dots, x_n]$ and

$$\tilde{F}_j = \frac{\prod_{i=1}^s F_i}{F_j} \text{ for } j = 1, \dots, s,$$

then

$$(\tilde{F}_1, \dots, \tilde{F}_s) = \bigcap_{1 \le i < j \le s} (F_i, F_j).$$

The variety $\mathbb{X}$ in $\mathbb{P}^n$ of the ideal

$$(\tilde{F}_1, \dots, \tilde{F}_s) = \bigcap_{1 \le i < j \le s} (F_i, F_j)$$

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is called a *star-configuration* in $\mathbb{P}^n$ of type s defined by general forms $F_1 \dots, F_s$. Furthermore, if $F_1, \dots, F_s$ are all general *linear* forms, then $\mathbb{X}$ is called a *linear star-configuration* of type s (see also [1, 2, 6, 7]).

In this paper, we discuss some applications of star-configurations in $\mathbb{P}^n$. In other words, we study some examples of secant varieties to the variety of reducible forms in $\mathbb{P}^2$, which is not defective, using the sum of ideals of two star-configurations in $\mathbb{P}^2$.

In Section 2, we discuss the Hilbert function of the ideal of the union of two star-configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^2$ when $\lambda = (1, \dots, 1, 3, \dots, 3)$, which we will use to find the dimension of secant varieties to the variety of reducible forms in Section 3 (see also [3, 4, 5]).

In Section 3, we prove that if $\lambda = (1, \dots, 1, 3, \dots, 3)$, then the secant variety $\mathrm{Sec}_1(\mathbb{X}_{\lambda,2})$ to the variety $\mathbb{X}_{\lambda,2}$ is not defective for $3 < d$ (see Theorem 3.5). Finally, we give a question on secant varieties for the further study.

2. The union of two star-configurations in $\mathbb{P}^2$ defined by linear forms and cubic forms

In this section, we study the Hilbert function and the minimal generators of the ideal the union of two star-configurations in $\mathbb{P}^2$, and we use these in the next section. Throughout this paper,

a solid line	——	$\mathbb{L}_i$	is a line defined by a linear form	L_i,
a dashed line	- - - -	$\mathbb{M}_i$	is a line defined by a linear form	M_i,
a thick line	**——**	$\mathbb{L}_i$	is a line defined by a cubic form	L_i^3,

for $1 \leq i \leq s$ with $s \geq 2$. Moreover, we define that

$P_{i,j}$ is a point defined by linear forms L_i, L_j,
$\boldsymbol{P}_{i,j}$ is a double point defined by a linear form and a quadratic form L_i, L_j^2,
$\mathcal{P}_{i,j}$ is a triple point defined by a linear form and a cubic form L_i, L_j^3,
$Q_{i,j}$ is a point defined by linear forms M_i, M_j, and
$\boldsymbol{Q}_{i,j}$ is a double point defined by a linear form and a quadratic form M_i, M_j^2,

where L_i, L_j and M_i, M_j are linear forms in R with $i < j$.

Let $\lambda = (d_1, \dots, d_s)$, where $1 \leq d_1 \leq \cdots \leq d_s$ and $d := \sum_{i=1}^{s} d_i$. We denote by $\mathbb{X}^{(\lambda)}$ a star-configuration in $\mathbb{P}^2$ defined by forms $F_1, \dots, F_s$ in $R = \Bbbk[x_0, x_1, x_2]$ with $\deg(F_i) = d_i$ for every i.

Lemma 2.1. *Let* $\lambda = (1, \ldots, 1, 3)$, *and* $\mathbb{X}_1 := \mathbb{X}_1^{(\lambda)}$ *and* $\mathbb{X}_2 := \mathbb{X}_2^{(\lambda)}$ *be star-configurations in* $\mathbb{P}^2$ *with* $5 \le d \le 7$. *Then* $\mathbb{X} := \mathbb{X}_1 \cup \mathbb{X}_2$ *has generic Hilbert function. In particular,* $\dim_{\Bbbk} \left(I_{\mathbb{X}_1} + I_{\mathbb{X}_2}\right)_d = \binom{d+2}{2}$.

Proof. First, we assume that $\mathbb{X}_1$ and $\mathbb{X}_2$ are defined by L_1, L_2, L_3^3, and $M_1, M_2, M_3M_4M_5$, respectively, where L_i and M_i are linear forms in R for every i (see Figure 1). Furthermore, we assume that L_1 vanishes on four points in $\mathbb{X}_1$, and one more point in $\mathbb{X}_2$, defined by two linear forms M_1 and M_2, and L_2 vanishes on three points in $\mathbb{X}_1$ and one more point in $\mathbb{X}_2$ defined by linear forms M_1 and M_5 (see Figure 1 again).

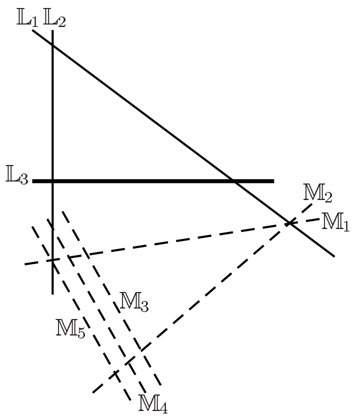

Figure 1

By *Bezóut*'s Theorem, for $N \in (I_{\mathbb{X}})_4$, $N = \alpha L_1 L_2 M_2 M_1$ for some $\alpha \in \Bbbk$. Therefore, the Hilbert function of $\mathbb{X}$ is $1 \quad 3 \quad 6 \quad 10 \quad 14 \quad \rightarrow$, as we wished.

Using the following exact sequence

$$0 \rightarrow R/I_{\mathbb{X}} \rightarrow R/I_{\mathbb{X}_1} \oplus R/I_{\mathbb{X}_2} \rightarrow R/(I_{\mathbb{X}_1} + I_{\mathbb{X}_2}) \rightarrow 0,$$

we have $\dim_{\Bbbk}(I_{\mathbb{X}_1} + I_{\mathbb{X}_2})_5 = \binom{5+2}{2}$. By the same method as above, one can show that $\mathbb{X}$ has generic Hilbert function when $d = 6, 7$, and so

$$\dim_{\Bbbk}(I_{\mathbb{X}_1} + I_{\mathbb{X}_2})_d = \binom{d+2}{2},$$

for $5 \le d \le 7$, which completes the proof. □

Theorem 2.2. *Let* $\lambda = (1, \ldots, 1, 3)$ *and let* $\mathbb{X}$ *be the union of two star-configurations* $\mathbb{X}_1 := \mathbb{X}_1^{(\lambda)}$ *and* $\mathbb{X}_2 := \mathbb{X}_2^{(\lambda)}$ *in* $\mathbb{P}^2$ *with* $d \ge 8$. *Then*

$$\dim_{\Bbbk}(I_{\mathbb{X}_1} + I_{\mathbb{X}_2})_d = 4d + 8.$$

Proof. First, we assume that $\mathbb{X}_1$ and $\mathbb{X}_2$ are defined by $L_1, \dots, L_{d-3}, L_{d-2}^3$ and $M_1, \dots, M_{d-3}, M_{d-2}M_{d-1}^2$, respectively, where L_i and M_j are linear forms in R for every i and j. Without loss of generality, we assume

$$\begin{array}{lll}
L_1 & \text{vanishes on } d+1 \text{ points} & P_{1,2}, \dots, P_{1,d-3}, \mathcal{P}_{1,d-2}, \boldsymbol{Q}_{1,d-1}, \\
L_2 & \text{vanishes on } d \text{ points} & P_{2,3}, \dots, P_{3,d-3}, \mathcal{P}_{2,d-2}, \boldsymbol{Q}_{2,d-1}, \\
 & \vdots & \\
L_{d-3} & \text{vanishes on 5 points} & \mathcal{P}_{d-3,d-2}, \boldsymbol{Q}_{d-3,d-1}.
\end{array}$$

By *Bezóut*'s Theorem, for $N \in (I_{\mathbb{X}})_d$, $N = L_1 \cdots L_{d-3}N'$ for some $N' \in R_3$. Since a linear star-configuration $\mathbb{Y}$ in $\mathbb{P}^2$ defined by $M_1, \dots, M_{d-2}$ has no generators in degree 3 and N' has to vanishes on all points in $\mathbb{Y}$, we see that $N' = 0$, i.e., $N = 0$, and so $\dim_{\Bbbk}(I_{\mathbb{X}})_d = 0$.

Using the following exact sequence

$$0 \quad \to \quad R/I_{\mathbb{X}} \quad \to \quad R/I_{\mathbb{X}_1} \oplus R/I_{\mathbb{X}_2} \quad \to \quad R/(I_{\mathbb{X}_1} + I_{\mathbb{X}_2}) \quad \to \quad 0,$$

we have

$$\dim_{\Bbbk}(I_{\mathbb{X}_1} + I_{\mathbb{X}_2})_d = 2 \cdot \dim_{\Bbbk} R_d - 2 \cdot \deg(\mathbb{X}_1) = 4d + 8,$$

which completes the proof of this theorem. $\square$

Lemma 2.3. *Let $\lambda = (1, 3, 3)$ or $(1, 1, 3, 3)$, and $\mathbb{X}_1 := \mathbb{X}_1^{(\lambda)}$ and $\mathbb{X}_2 := \mathbb{X}_2^{(\lambda)}$ be star-configurations in $\mathbb{P}^2$. Then $\mathbb{X} := \mathbb{X}_1 \cup \mathbb{X}_2$ has generic Hilbert function. In particular, $\dim_{\Bbbk}\left(I_{\mathbb{X}_1} + I_{\mathbb{X}_2}\right)_d = \binom{d+2}{2}$.*

Proof. We shall introduce only the proof for the case $\lambda = (1, 3, 3)$, and we omit the proof for the case $\lambda = (1, 1, 3, 3)$ since it simply reiterates the same arguments we will use. So we assume $\lambda = (1, 3, 3)$. Let $\lambda' = (3, 3)$, and $\mathbb{Y}_1 := \mathbb{X}_1^{(\lambda')}$ and $\mathbb{Y}_2 := \mathbb{X}_2^{(\lambda')}$ be star-configurations in $\mathbb{P}^2$. Let $\mathbb{Y}_1$ and $\mathbb{Y}_2$ be defined by $L_2L_3L_4, L_5L_6L_7$ and $M_2M_3M_4, M_5M_6M_7$, respectively, where L_i and M_j are linear forms in R for every i, j. Then it is not hard to see that the Hilbert function of $\mathbb{Y} := \mathbb{Y}_1 \cup \mathbb{Y}_2$ is $1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 18 \quad \to .$

Now assume that $\mathbb{X}_1 := \mathbb{X}_1^{(\lambda)}$ is defined by $L_1, L_2L_3L_4, L_5L_6L_7$, where L_1 is a linear form in R, and $\mathbb{Z} := \mathbb{X}_1 \cup \mathbb{Y}_2$. Using the following exact sequence

$$0 \quad \to \quad R/I_{\mathbb{Z}} \quad \to \quad R/I_{\mathbb{Y}} \oplus R/(L_1, G_6) \quad \to \quad R/(I_{\mathbb{Y}}, L_1, G_6) \quad \to \quad 0,$$

where $G_6 = L_2 \cdots L_7$, we obtain the following Hilbert functions.

$$\begin{array}{rcccccccccl}
\mathbf{H}(R/I_{\mathbb{Z}},-) & : & 1 & 3 & 6 & 10 & 15 & - & 24 & \to, \\
\mathbf{H}(R/I_{\mathbb{Y}},-) & : & 1 & 3 & 6 & 10 & 15 & 18 & 18 & \to, \\
\mathbf{H}(R/(L_1,G_6),-) & : & 1 & 2 & 3 & 4 & 5 & 6 & 6 & \to, \\
\mathbf{H}(R/(I_{\mathbb{Y}},L_1,G_6),-) & : & 1 & 2 & 3 & 4 & 5 & - & 0 & \to, \\
\mathbf{H}(R/(I_{\mathbb{Y}},L_1),-) & : & 1 & 2 & 3 & 4 & 5 & 3 & 0 & \to.
\end{array}$$

By *Bezóut*'s Theorem, it is easily to show that $(I_{\mathbb{Z}})_5 = \{0\}$, and so the Hilbert function of $\mathbb{Z}$ is $\mathbf{H}(R/I_{\mathbb{Z}},-) \;:\; 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 24 \quad \to,$ as we wished. Using the same idea as above and by *Bezóut*'s Theorem, one can show that $\mathbb{X}$ has generic Hilbert function. Therefore, we get that

$$\dim_{\Bbbk}\left(I_{\mathbb{X}_1} + I_{\mathbb{X}_2}\right)_d = \binom{d+2}{2},$$

for $d = 7, 8$, as we wished. □

By the same idea as in the proof of Theorem 2.2, the following theorem can be easily obtained, and so we omit the proof.

THEOREM 2.4. *Let* $\lambda = (\underbrace{1,\ldots,1}_{(s-\ell)\text{-times}},\underbrace{3,\ldots,3}_{\ell\text{-times}})$ *and let* $\mathbb{X}$ *be the union of two star-configurations* $\mathbb{X}_1 := \mathbb{X}_1^{(\lambda)}$ *and* $\mathbb{X}_2 := \mathbb{X}_2^{(\lambda)}$ *in* $\mathbb{P}^2$ *with either* $\ell \geq 3$ *or* $\ell = 2$ *and* $d \geq 9$. *Then* $\dim_{\Bbbk}(I_{\mathbb{X}_1} + I_{\mathbb{X}_2})_d = 4d + 6\ell + 2$.

3. Varieties of reducible forms and their secants

We first recall the definition of the secant variety $\mathrm{Sec}_{s-1}(\mathbb{X})$ to the variety $\mathbb{X}$ in $\mathbb{P}^n$. Let $\lambda \vdash d$ denote a *partition* of the integer d, i.e.

$$\lambda = (d_1,\ldots,d_s) \quad \text{where} \quad 1 \leq d_1 \leq \cdots \leq d_s \quad \text{and} \quad \textstyle\sum_{i=1}^{s} d_i = d.$$

We associate a variety, denoted by $\mathbb{X}_{\lambda,n}$, to $R = \Bbbk[x_0,x_1,\ldots,x_n]$ and λ, which is defined by

$$\mathbb{X}_{\lambda,n} := \{[F] \in \mathbb{P}(R_d) \mid F = F_1 \cdots F_s,\ \deg F_i = d_i\}.$$

Such varieties are called *varieties of reducible forms*. If λ is the d-tuple $(1,\ldots,1)$, then the variety is often referred to as the variety of *completely decomposable forms* or *split* forms. In this case, $\mathbb{X}_{\lambda,n}$ is denoted by $\mathrm{Split}_d(\mathbb{P}^n)$.

Since the map below has only finite fibers,

$$\mathbb{P}(R_{d_1}) \times \cdots \times \mathbb{P}(R_{d_s}) \longrightarrow \mathbb{X}_{\lambda,n}, \text{ where } [F_1] \times \cdots \times [F_s] \longrightarrow [F_1 \cdots F_s]$$

the dimension of $\mathbb{X}_{\lambda,n}$ is

$$\dim \mathbb{X}_{\lambda,n} = \left(\binom{d_1+n}{n} - 1\right) + \cdots + \left(\binom{d_s+n}{n} - 1\right) = \sum_{i=1}^{s} \binom{d_i+n}{n} - s.$$

DEFINITION 3.1. Let $\mathbb{X}_1, \ldots, \mathbb{X}_s$ all be non-degenerate, reduced and irreducible varieties in $\mathbb{P}^n$ with $\dim \mathbb{X}_i = d_i$.

(a) Choose points $P_i \in \mathbb{X}_i$ such that $\{P_1, \ldots, P_s\}$ are linearly independent (and so $s \leq n$). The *join* of $\{P_1, \ldots, P_s\}$ is the linear space spanned by the points, i.e.,

$$\Lambda(P_1, \ldots, P_s) := \langle P_1, \ldots, P_s \rangle \simeq \mathbb{P}^{s-1}.$$

(b) The *join* of $\mathbb{X}_1, \ldots, \mathbb{X}_s$ is $\Lambda(\mathbb{X}_1, \ldots, \mathbb{X}_s) :=$ $\overline{\bigcup\{\Lambda(P_1, \ldots, P_s) \mid \text{for } P_1, \ldots, P_s \text{ linearly independent}, P_i \in \mathbb{X}_i\}}$.

(c) If $\mathbb{X}_1 = \cdots = \mathbb{X}_s = \mathbb{X}$ with $\dim \mathbb{X} = d$, then we write $\Lambda(\mathbb{X}_1, \ldots, \mathbb{X}_s)$ $= \mathrm{Sec}_{s-1}(\mathbb{X})$ and call it the $(s-1)$-st *secant variety* to $\mathbb{X}$.

The number of parameters shows that the upper bound of the dimension of the join is

$$\dim \Lambda(\mathbb{X}_1, \ldots, \mathbb{X}_s) \leq \min\left\{n, \sum_{i=1}^{s} d_i + (s-1)\right\},$$

and thus

$$\dim \mathrm{Sec}_{s-1}(\mathbb{X}) \leq \min\{n, ds + (s-1)\}.$$

DEFINITION 3.2. Let $\mathbb{X} \subset \mathbb{P}^n$ be a projective variety of dimension d. Then the *expected dimension* of the secant variety $\mathrm{Sec}_{s-1}(\mathbb{X})$ to $\mathbb{X}$ is defined by

$$\mathrm{expdim}(\mathrm{Sec}_{s-1}(\mathbb{X})) = \min\{n, ds + (s-1)\}.$$

However, the expected dimension of $\mathrm{Sec}_{s-1}(\mathbb{X})$ is not always the same as $\dim \mathrm{Sec}_{s-1}(\mathbb{X})$. If $\mathrm{expdim}(\mathrm{Sec}_{s-1}(\mathbb{X})) - \dim \mathrm{Sec}_{s-1}(\mathbb{X}) > 0$, we say that the secant variety $\mathrm{Sec}_{s-1}(\mathbb{X})$ to $\mathbb{X}$ is *defective*.

Since we are interested in the secants to the varieties of reducible forms, we introduce another important result in [5] to find a description of the tangent space at a generic point of those varieties.

PROPOSITION 3.3 ([5]). *Let $\lambda \vdash d$, $\lambda = (d_1, \ldots, d_s)$ and let $\mathbb{X}_{\lambda,n} \subset \mathbb{P}^{\binom{d+n}{n}-1}$. Let $P = [F_1 \cdots F_s]$ be a generic point of $\mathbb{X}_{\lambda,n}$ where $\deg F_i = d_i$, $i = 1, \ldots, s$. Then $T_{P,\mathbb{X}_{\lambda,n}} = \mathbb{P}(V_P)$ where V_P is the subspace of $R_d = \Bbbk[x_0, \ldots, x_n]_d$ defined by $V_P := (\tilde{F}_1, \ldots, \tilde{F}_s)$, where $\tilde{F}_i = \frac{\prod_{j=1}^{s} F_j}{F_i}$ for every $i = 1, \ldots, s$.*

The following corollary is useful for finding whether or not the given secant varieties are defective.

COROLLARY 3.4 ([5]). *Let* $\lambda \vdash d$, $\lambda = (d_1, \ldots, d_s)$ *and let* $\mathbb{X}_{\lambda,n} \subset \mathbb{P}^{\binom{d+n}{n}-1}$. *Let* $P_1, \ldots, P_s$ *be* s *generic points on* $\mathbb{X}_{\lambda,n}$. *Then*

$$\dim \mathrm{Sec}_{s-1}(\mathbb{X}_{\lambda,n}) = \left[\binom{d+n}{n} - \mathbf{H}(A,d)\right] - 1 = \dim_{\Bbbk} I_d - 1$$

where $A = R/I$ *and* $I = \mathcal{T}_{P_1} + \cdots + \mathcal{T}_{P_s}$.

In this paper, we are interested in the secant variety $\mathrm{Sec}_1(\mathbb{X}_{\lambda,2})$ to the variety $\mathbb{X}_{\lambda,n} := \{[F] \in \mathbb{P}(R_d) \mid F = F_1 \cdots F_s,\ \deg F_i = 1 \text{ or } 2\}$.

In [3] and [6] the authors showed that the secant variety $\mathrm{Sec}_1(\mathbb{X}_{\lambda,n}) = \mathrm{Sec}_1(\mathrm{Split}_d(\mathbb{P}^n))$ is not defective for $n \geq 2$. Moreover, since it is not hard to show that the secant variety $\mathrm{Sec}_1(\mathbb{X}_{\lambda,2})$ is not defective when $d_i = 3$ for every i, we shall not introduce the proof in this paper. Thus we assume that $d_1 = \cdots = d_{s-\ell} = 1$ and $d_{s-\ell+1} = \cdots = d_s = 3$ with $1 \leq \ell < s$ for the rest of this paper. We now introduce the main theorem in this paper.

THEOREM 3.5. *Let* $\lambda \mapsto d$ *and* $\lambda = (\underbrace{1, \ldots, 1}_{(s-\ell)\text{-times}}, \underbrace{3, \ldots, 3}_{\ell\text{-times}})$. *Then the secant variety* $\mathrm{Sec}_1(\mathbb{X}_{\lambda,2})$ *is not defective for* $s \geq 3$ *and* $1 \leq \ell < s$.

Proof. If $d = 5$ and $\ell = 1$, then by Lemma 2.1 and Corollary 3.4,

$$\begin{aligned}
&\mathrm{expdimSec}_1(\mathbb{X}_{\lambda,2})\\
&= \min\left\{2 \cdot \dim((\mathbb{P}(R_1) \times \mathbb{P}(R_1) \times \mathbb{P}(R_1) \times \mathbb{P}(R_3)) + 1, \binom{5+2}{2} - 1\right\}\\
&= 20 = \dim_{\Bbbk}(I_{\mathbb{X}_1} + I_{\mathbb{X}_2})_5 - 1 = \dim \mathrm{Sec}_1(\mathbb{X}_{\lambda,2}).
\end{aligned}$$

By the same method as above with Lemmas 2.1, 2.3, and Corollary 3.4, one can see that $\mathrm{expdimSec}_1(\mathbb{X}_{\lambda,2}) = \dim \mathrm{Sec}_1(\mathbb{X}_{\lambda,2})$ for either $d = 6, 7$ and $\ell = 1$ or $d = 7, 8$ and $\ell = 2$.

Now suppose either $\ell = 1$ and $d \geq 8$ or $\ell = 2$ and $d \geq 9$. Then, by Theorems 2.2, 2.4 , and Corollary 3.4,

$$\begin{aligned}
&\mathrm{expdimSec}_1(\mathbb{X}_{\lambda,2})\\
&= \min\Big\{2 \cdot \dim((\underbrace{\mathbb{P}(R_1) \times \cdots \times \mathbb{P}(R_1)}_{(s-1)\text{-times}} \times \mathbb{P}(R_3)) + 1, \binom{d+2}{2} - 1\Big\}\\
&= 4d + 7 \text{ (since } d \geq 8)\\
&= \dim_{\Bbbk}(I_{\mathbb{X}_1} + I_{\mathbb{X}_2})_d - 1 = \dim \mathrm{Sec}_1(\mathbb{X}_{\lambda,2}),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{expdimSec}_1(\mathbb{X}_{\lambda,2})\\
&= \min\Big\{2 \cdot \dim((\underbrace{\mathbb{P}(R_1) \times \cdots \times \mathbb{P}(R_1)}_{(s-2)\text{-times}} \times \mathbb{P}(R_3) \times \mathbb{P}(R_3)) + 1, \binom{d+2}{2} - 1\Big\}\\
&= 4d + 13 \text{ (since } d \geq 8) = \dim_{\Bbbk}(I_{\mathbb{X}_1} + I_{\mathbb{X}_2})_d - 1 = \dim \mathrm{Sec}_1(\mathbb{X}_{\lambda,2}),
\end{aligned}$$

respectively, as we wished.

Now assume that $\ell \geq 3$. Then by Theorem 2.4 and Corollary 3.4,

$$\begin{aligned}
&\operatorname{expdimSec}_1(\mathbb{X}_{\lambda,2})\\
&= \min\left\{2 \cdot ((\underbrace{\mathbb{P}(R_1) \times \cdots \times \mathbb{P}(R_1)}_{(s-\ell)\text{-times}} \times \underbrace{\mathbb{P}(R_3) \times \cdots \times \mathbb{P}(R_3)}_{\ell\text{-times}}) + 1, \binom{d+2}{2} - 1\right\}\\
&= 4d + 6\ell + 1 \text{ (since } d \geq 3\ell + 1)\\
&= \dim_{\Bbbk}(I_{\mathbb{X}_1} + I_{\mathbb{X}_2})_d - 1\\
&= \dim \operatorname{Sec}_1(\mathbb{X}_{\lambda,2}),
\end{aligned}$$

which completes the proof. □

Now we give a question on secant varieties to the variety $\mathbb{X}_{\lambda,n}$.

Question 3.6. Is the secant variety $\operatorname{Sec}_{s-1}(\mathbb{X}_{\lambda,2})$ to the variety $\mathbb{X}_{\lambda,2}$ non-defective for $s > 2$ when $\lambda = (1, \ldots, 1, 3, \ldots, 3)$?

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